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ON REFINED CREEP BOUNDS AND BRITTLE DAMAGE ESTIMATES FOR PRESSURE VESSELS†

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Abstract—New bounds are obtained for stresses in pressure vessels subject to primary or secondary creep and including the effect of elastic strains. These results are applied to the estimation of the times of initiation of rupture using the Kachanov theory of brittle damage.

NOTATION

- a inner radius
- b outer radius
- E Young's modulus
- j constant 2 or 3 for a cylinder or sphere, respectively
- K creep constant
- n exponent in the Power Law
- $p(t)$ pressure
- $P(t)$ related to pressure by $P(t) = \left(\frac{1}{2}\right) 3^{(3-j)/2} \cdot p(t)$
- r radius
- R constant $\equiv \left(\frac{a}{b}\right)^j$
- s_{ij} stress deviator
- t time
- ϵ_{ij} strain tensor
- μ constant $\equiv EK$ for secondary creep
- ν damage constant (Kachanov)
- σ_{ij} stress tensor
- σ_{max} maximum principal stress
- σ_r radial stress
- σ_θ circumferential stress
- σ effective stress
- $\sigma_{cB}^{(5)}$ stress causing creep rupture in 10^5 hr

1. INTRODUCTION

In [1], the boundary value problems for primary creep in either cylindrical or spherical pressure vessels subject to a non-decreasing internal pressure were reduced to the following integral equation

$$\sigma(r, t) = \frac{\beta P(t)}{r^j} + \mu \left(\frac{\beta}{r^j} \int_a^b \left[\int_0^t \sigma^n(\xi, \tau)\, d\tau \right]^{1/(m+1)} \frac{d\xi}{\xi} - \left[\int_0^t \sigma^n(r, \tau)\, d\tau \right]^{1/(m+1)} \right). \tag{1.1}$$

Here, the unknown function $\sigma(r, t)$ is the so-called "effective stress" at radial or axial distance r from the center of the vessel and time t. $P(t)$ is related to internal pressure by

$$P(t) = \left(\frac{1}{2}\right) 3^{(3-j)/2} \cdot p(t), \tag{1.2}$$

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a and b are the internal and external radii of the vessel, and

$$\beta^{-1} \equiv \int_a^b \frac{d\xi}{\xi^{j+1}}, \tag{1.3}$$

where $j = 2$ for cylinders, $j = 3$ for spheres. When $m \neq 0$ we have the case of primary creep.

It was also shown in [1] that $\sigma(r, t)$ satisfies the inequalities

$$\frac{\beta P(t)}{b^j} \leq \sigma(r, t) \leq \frac{\beta P(t)}{a^j} \quad (a \leq r \leq b, t \geq 0). \tag{1.4}$$

It was suggested in [1] that the above bounds might, themselves, be used as the basis for more refined bounds. This paper presents a first attempt at this. Such bounds can be of great practical importance to the designer, giving a quick check on preliminary designs, thus saving the cost of expensive computer solutions. Even if more accurate computer solutions are necessary, the bounds can be used as a check on these solutions.

In Section 2 a set of bounds is derived that reduce to the exact solution for $t = 0$. They are shown to vary from the initial elastic solution as $t^{(1/m+1)}$ and thus for "short" times are superior to the bounds given by (1.4). In Section 3, we consider only the case of secondary creep, and derive bounds which for large time converge to a limiting stress, $\sigma(r, \infty)$, as given in [2] as

$$\sigma(r, \infty) = \frac{j}{n} P(\infty)[a^{-j/n} - b^{-j/n}]^{-1} r^{-j/n}. \tag{1.5}$$

In Section 4 we use the results of Sections 2 and 3 to make predictions concerning brittle creep rupture in the sense of Kachanov [3]. We present a criterion for determining when failure will be initiated at a place other than the outer surface of the vessel. Also assuming failure is initiated at the outer surface, we obtain upper and lower bounds on the time at which failure starts.

The estimates derived in this paper are applied to real metals, using physical constants tabulated by Odqvist in [4].

2. SHORT TIME BEHAVIOR OF EFFECTIVE STRESS

Using the previously derived bounds (1.4) we are able to derive the improved bounds given by

Theorem 2.1. *For a symmetric pressure vessel undergoing primary creep with non-decreasing internal pressure proportional to* P(t),

$$\left|\sigma(r, t) - \frac{\beta P(t)}{r^j}\right| \leq m(r, t), \tag{2.1}$$

where

$$m(r, t) = \frac{\mu}{r^j}\left(\beta^n \int_0^t P^n(\tau)\, d\tau\right)^{1/(m+1)} (a^{j(1-(n/(m+1)))} - b^{j(1-(n/(m+1)))}). \tag{2.2}$$

Proof: Define

$$\phi(r, t) = \int_0^t \sigma^n(r, \tau)\, d\tau. \tag{2.3}$$

Equation (1.1) may thus be written

$$\sigma(r, t) = \frac{\beta P(t)}{r^j} + \mu\left(\frac{\beta}{r^j}\int_a^b \phi^{1/(m+1)}(\xi, t)\frac{d\xi}{\xi} - \phi^{1/(m+1)}(r, t)\right). \tag{2.4}$$

In [1] it was shown that

$$\frac{\partial}{\partial r}(r^j \phi^{1/(m+1)}(r, t)) \leq 0. \quad (t > 0) \tag{2.5}$$

Applying eqn (2.5) to eqn (2.4) and using (1.3) we get

$$\sigma(r,t) \leqslant \frac{\beta P(t)}{r^j} + \frac{\mu}{r^j}[a^j \phi^{1/(m+1)}(a,t) - b^j \phi^{1/(m+1)}(b,t)], \tag{2.6}$$

and

$$\sigma(r,t) \geqslant \frac{\beta P(t)}{r^j} - \frac{\mu}{r^j}[a^j \phi^{1/(m+1)}(a,t) - b^j \phi^{1/(m+1)}(b,t)]. \tag{2.7}$$

Putting eqn (1.4) into eqn (2.4) gives

$$\phi^{1/(m+1)}(a,t) \leqslant \left(\frac{\beta^n}{a^{nj}} \int_0^t P^n(\tau)\,d\tau\right)^{1/(m+1)}, \tag{2.8}$$

and

$$\phi^{1/(m+1)}(b,t) \geqslant \left(\frac{\beta^n}{b^{nj}} \int_0^t P^n(\tau)\,d\tau\right)^{1/(m+1)}. \tag{2.9}$$

Substituting eqns (2.8) and (2.9) into (2.6) yields

$$\sigma(r,t) \leqslant \frac{\beta P(t)}{r^j} + \frac{\mu}{r^j}\left(\beta^n \int_0^t P^n(\tau)\,d\tau\right)^{1/(m+1)} \left(\frac{1}{a^{j(n/(m+1)-1)}} - \frac{1}{b^{j(n/(m+1)-1)}}\right).$$

Similarly from eqns (2.7) to (2.9) we get

$$\sigma(r,t) \geqslant \frac{\beta P(t)}{r^j} - \frac{\mu}{r^j}\left(\beta^n \int_0^t P^n(\tau)\,d\tau\right)^{1/(m+1)} (a^{j(1-(n/[m+1]))} - b^{j(1-(n/[m+1]))}),$$

and the theorem is proved.

For the case when $P(t) \equiv P$, a constant, eqns (2.1) and (2.2) can be written:

$$\left|\sigma(r,t) - \frac{\beta P}{r^j}\right| \leqslant m(r) t^{1/(m+1)}, \tag{2.10}$$

$$m(r) = \frac{\mu}{r^j}(\beta P)^{n/(m+1)}(a^{j(1-(n/[m+1]))} - b^{j(1-(n/[m+1]))}). \tag{2.11}$$

Thus we see that, for the case of constant pressure, the bounds on $\sigma(r,t)$ are a variation from the initial elastic response,

$$\sigma(r,0) = \frac{\beta P(0)}{r^j}, \tag{2.12}$$

with the $m+1$ root of t.

It is of interest to note the values of t for which these bounds are superior to those given by eqn (1.4). Consider the case of constant pressure and let $r=b$. Equation (2.10) gives

$$\sigma(b,t) \leqslant \frac{\beta P}{b^j} + \mu(\beta P)^{n/(m+1)}(b^{-j} a^{j(1-(n/[m+1]))} - b^{-jn/(m+1)}) t^{1/(m+1)}. \tag{2.13}$$

Equation (2.13) is a better upper bound on $\sigma(r,t)$ for all times less than t^*, where t^* satisfies

$$\frac{\beta P}{a^j} = \frac{\beta P}{b^j} + \mu(\beta P)^{n/(m+1)}(b^{-j} a^{j(1-(n/[m+1]))} - b^{-jn/(m+1)}) t^{*1/(m+1)}. \tag{2.14}$$

Solving for t^* and using (1.3) and

$$R \equiv \left(\frac{a}{b}\right)^j, \tag{2.15}$$

we have

$$t^* = \frac{1}{(jP)^{n-m-1}\mu^{m+1}} \frac{(1-R)^n}{(R - R^{n/(m+1)})^{m+i}}. \tag{2.16}$$

Consider a hollow incompressible cylinder of 12% Cr steel at 850°F. Hult in [5] asserts that $K = 0.5 \times 10^{-20}$, $n = 7.5$, $m = 1.8$ and $E = 16{,}200$ Kg/mm^2. The values of t^* for various values of P and (b/a) have been tabulated in Table 1.

Table 1. Time when short time bounds equal constant bounds

Pressure Kg/mm^2 \ $\frac{b}{a}$	1.3	1.5	1.7	2.0
5	108	1238	6200	34,237
10	4.14	48	239	1317

For the case of secondary creep we consider eqn (1.1) with $m = 0$:

$$\sigma(r, t) = \frac{\beta P(t)}{r^j} + \mu\left(\frac{\beta}{r^j}\int_a^b \int_0^t \sigma^n(\xi, \tau)\,d\tau \frac{d\xi}{\xi} - \int_0^t \sigma^n(r, \tau)\,d\tau\right). \tag{2.17}$$

The following Corollary is immediate.

Corollary 2.1. *For a symmetric pressure vessel undergoing secondary creep with non-decreasing internal pressure proportional to* P(t)

$$\left|\sigma(r, t) - \frac{\beta P(t)}{r^j}\right| \leq m(r, t), \tag{2.18}$$

where

$$m(r, t) = \frac{\mu\beta^n}{r^j}\int_0^t P^n(\tau)\,d\tau(a^{j(1-n)} - b^{j(1-n)}). \tag{2.19}$$

Note that at $t = 0$ the bounds (2.18) reduce to the exact solution. In this sense they significantly improve on the previously obtained bounds (1.4). In the case where the pressure, P, is a constant function of t, for $t > 0$, the bounds on $\sigma(r, t)$ take the form

$$\left|\sigma(r, t) - \frac{\beta P}{r^j}\right| \leq m(r)t, \tag{2.20}$$

where

$$m(r) = \frac{\mu}{r^j}\beta^n P^n(a^{j(1-n)} - b^{j(1-n)})$$
$$= \mu\left(\frac{a}{r}\right)^j P^n j^n \frac{1 - R^{n-1}}{(1-R)^n}. \tag{2.21}$$

Thus in the case of constant pressure, these bounds evolve away from the initial elastic response (2.12) linearly in time.

In order to understand how they relate to the previous bounds (1.4), we consider the case $P \equiv \text{constant}$ ($t > 0$) and $r = b$. In this case, eqn (2.20) becomes

$$\frac{\beta P}{b^j} - m(b)t \le \sigma(b, t) \le \frac{\beta P}{b^j} + m(b)t. \tag{2.22}$$

Clearly the lower bound for $\sigma(b, t)$ furnished by (1.4) is superior to that given above, but (2.22) does give a superior upper bound for all time t up to a time t^* given by

$$\frac{\beta P}{a^j} = \frac{\beta P}{b^j} + m(b)t^*. \tag{2.23}$$

Solving for t^* and using (2.21) gives

$$t^* = b^j \left[\frac{1}{a^j} - \frac{1}{b^j}\right][(\beta P)^{n-1}(a^{-(n-1)j} - b^{-(n-1)j})]^{-1}. \tag{2.24}$$

Using eqns (1.3) and (2.15) we have

$$t^* = \frac{(1-R)^n}{\mu(jP)^{n-1}R(1-R^{n-1})}. \tag{2.25}$$

Since μ is proportional to the creep constant K, (2.25) shows that the smaller K is, the larger the value of t^*. This relationship is reasonable, since the smaller the value of K, the less is the creep effect for fixed n, so that the stress redistribution is slower.

Since the function

$$f(R) = \frac{(1-R)^n}{R(1-R^{n-1})}$$

is decreasing on $0 \le R \le 1$ and $n \ge 2$, smaller values of R also give larger values of t^*. Again this is plausible since, in a thicker vessel, it should take a longer time for the redistribution of stresses to percolate from the loaded inner surface to the unloaded outer surface.

The values of t^* for various metals are given in Table 2.

Table 2. Comparison of bounds for secondary creep†

Material	Temp. (°C)	$\mu = EK$	n	t^* (hr.)
Carbon steel (cast)	455	9.695×10^{-8}	5	1,742
Carbon steel (rolled)	450	1.07×10^{-7}	5	1,577
	500	2.72×10^{-5}	3.3	413
Low alloy steel	450	2.97×10^{-10}	6	49,072
	500	1.73×10^{-8}	5.4	3,667
	550	5.4×10^{-6}	4.15	253
Chromium steel (forged)	450	4.602×10^{-11}	6.3	151,937
	500	1.19×10^{-8}	5.27	7,321
	550	9.3×10^{-7}	4.4	793
Nimonic 75 (forged)	650	8.874×10^{-6}	2.73	5,392

†Data is given for a cylindrical pressure vessel with $(b/a) = 2$ and $P = 5$ kg/mm^2.

3. LONG TIME BOUNDS FOR SECONDARY CREEP

It was shown in[2] that the effective stress in a symmetric pressure vessel undergoing secondary creep approaches a limit $\sigma(r,\infty)$ uniformly in r which is given by

$$\sigma(r,\infty)=\lim_{t\to\infty}\sigma(r,t)=\frac{j}{n}P(\infty)[a^{-j/n}-b^{-j/n}]^{-1}r^{-j/n}. \tag{3.1}$$

Clearly, a natural way to estimate the long time behavior of $\sigma(r,t)$ would be to bound the difference between $\sigma(r,t)$ and $\sigma(r,\infty)$; that is, we would like to find a function $f(r,t)$ such that

$$\lim_{t\to\infty}f(r,t)=0,\ r\in[a,b]$$

uniformly in r, and

$$|\sigma(r,t)-\sigma(r,\infty)|\leq f(r,t),\ r\in[a,b], \tag{3.2}$$

where eqn (3.2) either holds for all t or at least for $t>T$ where T is a known constant. It would also be of interest to note how this bound compares with our previous bounds.

To this end, we define the inner product

$$(v,w)=\beta\int_a^b v(\xi)w(\xi)\frac{d\xi}{\xi^{j+1}} \tag{3.3}$$

with the corresponding norm

$$\|v\|^2=\beta\int_a^b v^2(\xi)\frac{d\xi}{\xi^{j+1}},$$

and the linear functional $l(v)$ by

$$l(v)=\beta\int_a^b v(\xi)\frac{d\xi}{\xi^{j+1}}. \tag{3.4}$$

Notice that for any integrable functions v and w, and any constant C,

$$l(v)=(v,1),\quad l(vw)=(v,w),\quad l(C)=C,\quad (C,v-l(v))=0. \tag{3.5}$$

Our main result for large times, which, for the sake of simplicity only, is restricted to the case of constant pressure is given by

Theorem 3.1. *Let* $\sigma(r,t)$ *be the effective stress history corresponding to a pressure* p *which is constant for* $t>0$. *Then for all* $t>0$,

$$|\sigma(r,t)-\sigma(r,\infty)|\leq\sigma(r,0)(K_1+K_2t)\,e^{-Ct} \tag{3.6}$$

where

$$K_1=\frac{1}{n^2(1-R)R^{n-1}}\left\{R^n-nR+(n-1)+\frac{1}{R^{n-1}}\left(\frac{A}{2n-1}\right)^{1/2}\right\},$$

$$K_2=\frac{\mu}{n(1-R)^n}\left(\frac{Pj}{R}\right)^{n-1}\left(\frac{A}{2n-1}\right)^{1/2},$$

$$C=\mu n\left(\frac{jPR}{1-R}\right)^{n-1}=\mu n\left(\frac{\beta P}{b^j}\right)^{n-1},$$

$$A=(n-1)^2R^{2n}-n^2R^{2n-1}+2(2n-1)R^n-n^2R+(n-1)^2,$$

$$R=\left(\frac{a}{b}\right)^j.$$

Proof. Setting $m = 0$ in (1.1), we obtain the following equation for secondary creep:

$$\sigma(r,t) = \frac{\beta P}{r^j} + \mu\left(\frac{\beta}{r^j}\int_a^b \int_0^t \sigma^n(\xi,\tau)\mathrm{d}\tau \frac{\mathrm{d}\xi}{\xi} - \int_0^t \sigma^n(r,\tau)\,\mathrm{d}\tau\right). \tag{3.7}$$

Differentiated with respect to time this becomes

$$\dot{\sigma}(r,t) = \mu\left(\frac{\beta}{r^j}\int_a^b \sigma^n(\xi,t)\frac{\mathrm{d}\xi}{\xi} - \sigma^n(r,t)\right). \tag{3.8}$$

Define
$$w(r,t) \equiv r^j\sigma^n(r,t). \tag{3.9}$$

Then multiplication of eqn (3.8) by $r^j n\sigma^{n-1}(r,t)$ yields

$$\dot{w}(r,t) = \mu n\sigma^{n-1}(r,t)\left(\beta\int_a^b w(\xi,t)\frac{\mathrm{d}\xi}{\xi^{j+1}} - w(r,t)\right). \tag{3.10}$$

In the notation of (3.5) this takes the form

$$\dot{w} = \mu n\sigma^{n-1}(l(w) - w). \tag{3.11}$$

Due to (3.1), w_∞, the limit of $w(r,t)$ as $t \to \infty$, has the form

$$\begin{aligned} w_\infty &= \lim_{t\to\infty} r^j\sigma^n \\ &= \left(\frac{j}{n}P(\infty)\right)^n [a^{-j/n} - b^{-j/n}]^{-n}, \end{aligned}$$

i.e. w_∞ is a constant function of r. Therefore, by (3.5),

$$l(w_\infty) = w_\infty. \tag{3.12}$$

It is convenient in the derivation of (3.6) to first bound the quantity

$$v(r,t) \equiv w(r,t) - w_\infty. \tag{3.13}$$

For this purpose, we use (3.12) to rewrite (3.11) in the form

$$\dot{v} = \mu n\sigma^{n-1}(l(v) - v), \tag{3.14}$$

from which it follows that

$$\dot{v} + \mu n\sigma^{n-1}v = \mu n\sigma^{n-1}l(v). \tag{3.15}$$

If we treat (3.15) as a first order, linear, ordinary differential equation in v, we may solve easily by multiplying both sides by the integrating factor

$$\exp\left[\int_0^t \mu n\sigma^{n-1}(r,\tau)\,\mathrm{d}\tau\right],$$

and integrating to obtain

$$\begin{aligned} v(t)\exp\left[\mu n\int_0^t \sigma^{n-1}\,\mathrm{d}\tau\right] &= v(0) + \int_0^t \mu n\sigma^{n-1}l(v)\exp\left[\mu n\int_0^\tau \sigma^{n-1}\,\mathrm{d}\lambda\right]\mathrm{d}\tau \\ &= v(0) + \int_0^t l(v)\frac{\partial}{\partial\tau}\left(\exp\left[\mu n\int_0^\tau \sigma^{n-1}\,\mathrm{d}\lambda\right]\right)\mathrm{d}\tau. \end{aligned} \tag{3.16}$$

Integrating by parts now on the right-hand side of (3.16) and dividing both sides of the resulting equation by the exponential factor, we get

$$v(t) = (v(0) - lv(0)) \exp\left[-\mu n \int_0^t \sigma^{n-1}\, d\tau \right] + lv(t)$$
$$- \int_0^t l\dot{v}(\tau) \exp\left[-\mu n \int_\tau^t \sigma^{n-1}\, d\lambda \right] d\tau. \tag{3.17}$$

This equation shows that a bound for $|v(t)|$ will follow provided we bound $|lv|$ and $|l\dot{v}|$. However, since (3.13) implies that

$$v(r, \infty) = 0,$$

it follows that

$$lv(t) = lv(t) - lv(\infty) = -\int_t^\infty l\dot{v}(\tau)\, d\tau. \tag{3.18}$$

Equations (3.17) and (3.18) and the inequality (1.4) imply that

$$|v(t)| \leq |v(0) - lv(0)| e^{-Ct} + \int_t^\infty |l\dot{v}(\tau)| d\tau$$
$$+ \int_0^t |l\dot{v}(\tau)| e^{-C(t-\tau)}\, d\tau \tag{3.19}$$

where C is as defined in the statement of Theorem 3.1. Thus the problem is reduced to that of bounding $|l\dot{v}|$.

For this purpose we apply the operator l to both sides of (3.14) and use (3.5) plus Schwarz's inequality to see that

$$l\dot{v} = \mu n(\sigma^{n-1}, lv - v),$$
$$|l\dot{v}| \leq \mu n \|\sigma^{n-1}\| \|lv - v\| \leq \frac{C}{R^{n-1}} \|lv - v\|. \tag{3.20}$$

It remains to bound $\|lv - v\|$. Following a line of reasoning originated by Einarsson [6, 7], we use (3.5) and (3.14) to make the computation

$$\frac{d}{dt}(\|lv - v\|^2) = \frac{d}{dt}(lv - v, lv - v)$$
$$= 2(l\dot{v} - \dot{v}, lv - v)$$
$$= -2(\dot{v}, lv - v)$$
$$= -2\mu n(\sigma^{n-1}, [lv - v]^2),$$

so that

$$\frac{d}{dt}(\|lv - v)\|^2) \leq -2\mu n(\min_{[a,b]} \sigma^{n-1}) \|lv - v\|^2$$
$$\leq -2C\|lv - v\|^2.$$

This inequality can be integrated to obtain

$$\|lv - v\|(t) \leq \|lv - v\|(0)\, e^{-Ct}. \tag{3.21}$$

If the bound on $|l\dot{v}|$ which is implied by (3.20) and (3.21) is substituted into the right-hand side of (3.19) the result is

$$|v(t)| \leq [|v(0) - lv(0)| + R^{-n+1}\|lv - v\|(0)(1 + Ct)]\, e^{-Ct}. \tag{3.22}$$

In order to derive from (3.22) an estimate for $|\sigma(r, t) - \sigma(r, \infty)|$, we use the elementary fact that if

$$0 < x_0 \leq \min\{x_1, x_2\}, \qquad n \geq 1,$$

then

$$|x_2 - x_1| \leq \frac{1}{n x_0^{n-1}} |x_2^n - x_1^n|. \tag{3.23}$$

Since, by (1.4)

$$0 < \frac{\beta P}{b^j} \leq \min\{\sigma(r, t), \sigma(r, \infty)\},$$

we can use (3.9), (3.13), (3.22) and (3.23) to get

$$|\sigma(r, t) - \sigma(r, \infty)| \leq \frac{\mu}{r^j C} [|v(0) - lv(0)| + R^{-n+1}\|v - lv\|(0)(1 + Ct)]\, e^{-Ct}. \tag{3.24}$$

This inequality is essentially (3.6). All that remains is the straightforward but laborious computation of

$$\|lv - v\|(0) = \frac{(jP)^n a^j}{n(1 - R)^{n+1}} \left(\frac{A}{2n - 1}\right)^{1/2},$$

where $A = (n-1)^2 R^{2n} - n^2 R^{2n-1} + 2(2n-1)R^n - n^2 R + (n-1)^2$, and

$$|v(0) - lv(0)| = \frac{(jP)^n a^j}{n(1 - R)^{n+1}} \left|(1 - R^n) - \left(\frac{a}{r}\right)^{j(n-1)} (1 - R)\right|$$
$$\leq \frac{(jP)^n a^j}{n(1 - R)^{n+1}} (R^n - nR + n - 1).$$

This completes the proof.

From (3.6) we get the following exponential bounds on $\sigma(r, t)$:

$$\sigma(r, t) \leq \sigma(r, \infty) + \sigma(r, 0)(K_1 + K_2 t)\, e^{-Ct}, \tag{3.25a}$$

$$\sigma(r, t) \geq \sigma(r, \infty) - \sigma(r, 0)(K_1 + K_2 t)\, e^{-Ct}. \tag{3.25b}$$

Since, for $P > 0$ and $a < b$, it follows from (3.1) that

$$\frac{\beta P}{b^j} < \sigma(r, \infty) < \frac{\beta P}{a^j}, \tag{3.26}$$

there must exist a time t^* such that for $t > t^*$ the bounds (3.25) will give a better estimate of $\sigma(r, t)$ than (1.4).

Consider the effective stress at the outer surface of the pressure vessel. At $r = b$, (3.25a) becomes

$$\sigma(b, t) \leq \sigma(b, \infty) + \sigma(b, 0)(K_1 + K_2 t)\, e^{-Ct}.$$

To find t^* we must solve the transcendental equation

$$\sigma(a, 0) = \sigma(b, \infty) + \sigma(b, 0)(K_1 + K_2 t)\, e^{-Ct}.$$

Table 3. Comparison of long time bounds and constant bounds

Material	Temp. (°C)	n	μ = KE	Times t^* $\frac{b}{a}$ = 1.1	$\frac{b}{a}$ = 1.5	$\frac{b}{a}$ = 2.0
Low alloy steel (rolled)	450	6	2.97×10^{-10}	.249	8,090	1.1×10^{6}
	500	5.4	1.73×10^{-8}	.056	646	52,431
	550	4.15	5.4×10^{-6}	.032	47	1,319
	600	2.74	5.94×10^{-4}	0	9	90
	650	2.1	7.722×10^{-3}	(-.782)	(-2.1)	5.4
Chromium steel (forged)	450	6.3	4.602×10^{-11}	.439	24,104	4.3×10^{6}
	500	5.27	1.19×10^{-8}	.142	977	94,563
	550	4.4	9.3×10^{-7}	.079	149	5,115
	600	3.8	4.3×10^{-5}	.015	14	289
Nimonic 75 (forged)	650	2.73	8.87×10^{-6}	(-4.18)	619	6,084
Aluminum alloy 24S-T4	190	5.3	2.0636×10^{-8}	0	698	51,930
Aluminum alloy RR59	200	3.7	3.0351×10^{-6}	.309	249	4,765

The data has been tabulated for a cylindrical pressure vessel with an internal pressure of 10 kg/mm². This involves the approximating assumption of incompressibility.

The values of t^* for various metals have been tabulated in Table 3.

From the table we see that the exponential bounds are superior when the vessel is thin and the creep constant, K, is large. This is plausible since the exponential bounds estimate the difference between $\sigma(r,t)$ and the redistributed state $\sigma(r,\infty)$. For a thinner vessel with large K the stress should redistribute faster, so that $\sigma(r,t)$ will approach its steady state value faster.

Note that in some cases, especially for metals at very high temperatures, the exponential bound is a better bound for all time. In fact, for a low alloy steel at 650°C, even when the ratio of b to a is 2.0, the exponential bound is superior for all but the first 5.4 hr.

4. DAMAGE ESTIMATES FOR SECONDARY CREEP

Kachanov in[3] describes a theory of brittle creep rupture involving the use of a function $\psi(\underset{\sim}{x},t)$ which he calls the "continuity" function. This function indicates the deterioration of the material at a given point $\underset{\sim}{x}$ in the body at time t. When $t=0$, $\psi=1$. As t increases, the value of ψ decreases until at time $t=t_R$, $\psi(\underset{\sim}{x},t_R)=0$ and the material at $\underset{\sim}{x}$ is no longer able to carry a load. At such points, a failure front develops which moves through the material until the total carrying capacity of the structure is exhausted and collapse occurs.

Kachanov assumes that ψ is related to the maximum principal stress, $\sigma_{\max}$, through the differential equation

$$\frac{d\psi}{dt}=-C\left(\frac{\sigma_{\max}}{\psi}\right)^{\nu}, \tag{4.1}$$

where C and ν are material constants. Multiplication of (4.1) by ψ^{ν} and integration from 0 to t gives

$$1-\psi^{\nu+1}=C(1+\nu)\int_0^t\sigma_{\max}^{\nu}\,d\tau. \tag{4.2}$$

At the time of rupture, t_R, we note that $\psi(t_R)=0$ so that

$$1=C(1+\nu)\int_0^{t_R}\sigma_{\max}^{\nu}\,d\tau.$$

Tabulated data is usually not presented in the form of ν and C. What is given is $\sigma_{CB}^{(5)}$, the constant stress which produces creep rupture in 10^5 hr in a uniaxial creep rupture test. Thus,

$$1 = C(1+\nu)\sigma_{cB}^{(5)\nu} 10^5 \tag{4.3}$$

and

$$[\sigma_{cB}^{(5)}]^{\nu} 10^5 = \int_0^{t_K} \sigma_{\max}^{\nu}\, d\tau. \tag{4.4}$$

Also if σ_k is the constant stress needed for creep rupture in time t_k, we have

$$C(1+\nu)\sigma_k{}^{\nu} t_k = 1. \tag{4.5}$$

Therefore,

$$\left(\frac{\sigma_k}{\sigma_{cB}^{(5)}}\right)^{\nu} t_k = 10^5. \tag{4.6}$$

By finding t_k for various values of σ_k and making a log–log plot, the value of ν can be determined. Kachanov in[3] has found that for numerous structural steels, $\nu \approx 0.7n$, and that, in general, $\nu \leq n$ where n is the power in the Norton Power Law.

In his paper[3], Kachanov considers creep rupture of a thick-walled cylindrical tube assuming a stationary stress distribution corresponding to a state of plane secondary creep. The radial and tangential stress components are, respectively,

$$\sigma_r = s\left[1 - \left(\frac{b}{r}\right)^{(2/n)}\right], \tag{4.7}$$

$$\sigma_\theta = s\left[1 + \frac{2-n}{n}\left(\frac{b}{r}\right)^{(2/n)}\right], \tag{4.8}$$

$$s = p\left(\left[\frac{b}{a}\right]^{(2/n)} - 1\right)^{-1}. \tag{4.9}$$

Since all shear stresses are zero, σ_θ is the maximum principal stress and, for $n > 2, \sigma_\theta$ reaches its maximum at $r = b$, the outer surface.

However, as pointed out by Odqvist and Erikson[8], this need not be true if the constitutive law for the material includes both elastic and creep strains. In this case, the initial stress distribution is elastic, and, in the elastic problem, the maximum value of σ_θ occurs on the inner surface, $r = a$ (see 4.22 below). As t increases, the creep effect causes a redistribution of stress to the outer surface. However, if the material is extremely brittle or the vessel is very thick, a zone of damage may develop before the redistribution is complete. In this case, the locus of initial damage may be at the inner surface or somewhere in the interior.

It is our purpose in this section to apply the above developed bounds to such questions as whether or not damage first occurs on the outer surface and the estimation of the time when damage first occurs in the body.

First consider the case of an incompressible cylinder. In this case, the effective stress is given by

$$\sigma = \frac{\sqrt{3}}{2}(\sigma_\theta - \sigma_r). \tag{4.10}$$

Using this along with the quasi-static stress equation of motion

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = 0, \tag{4.11}$$

the boundary conditions

$$\sigma_r(a,t) = -p, \quad \sigma_r(b,t) = 0, \tag{4.12}$$

and the compatibility condition

$$\int_a^b \frac{\sigma(r,t)}{r}\,\mathrm{d}r = \left(\frac{\sqrt{3}}{2}\right)p \equiv P \tag{4.13}$$

we obtain

$$\sigma_r(r,t) = \frac{2}{\sqrt{3}}\int_a^r \sigma(\xi,t)\frac{\mathrm{d}\xi}{\xi} - p = -\frac{2}{\sqrt{3}}\int_r^b \sigma(\xi,t)\frac{\mathrm{d}\xi}{\xi}, \tag{4.14}$$

$$\sigma_\theta(r,t) = \frac{2}{\sqrt{3}}\left(\sigma(r,t) - \int_r^b \sigma(\xi,t)\frac{\mathrm{d}\xi}{\xi}\right). \tag{4.15}$$

From (1.5) with $j = 2$, and (1.2) we have

$$\sigma(r,\infty) = \lim_{t\to\infty}\sigma(r,t) = \left(\frac{\sqrt{3}}{n}\right)p\,[a^{(-2/n)} - b^{(-2/n)}]^{-1}r^{-2/n}. \tag{4.16}$$

Substituting from (4.16) into eqns (4.14) and (4.15) and integrating we find that they immediately reduce to (4.7) and (4.8). Thus the equations used by Kachanov are those for which the stress has completed redistribution.

The equations for an internally loaded hollow sphere corresponding to (4.14) and (4.15) are

$$\sigma_\phi(r,t) = \sigma_\theta(r,t) = \sigma(r,t) - 2\int_r^b \sigma(\xi,t)\frac{\mathrm{d}\xi}{\xi}, \tag{4.17}$$

$$\sigma_r(r,t) = 2\int_a^r \sigma(\xi,t)\frac{\mathrm{d}\xi}{\xi} - p. \tag{4.18}$$

Returning to the cylinder, we have at $t = 0$ the initial elastic response (2.12 with $j = 2$)

$$\sigma(r,0) = \frac{\beta P}{r^2} = \frac{\beta}{r^2}\left(\frac{\sqrt{3}}{2}p\right). \tag{4.19}$$

Substituting this into eqns (4.14) and (4.15), we get

$$\sigma_r(r,0) = p\left(\frac{\beta}{2}[a^{-2} - r^{-2}] - 1\right), \tag{4.20}$$

$$\sigma_\theta(r,0) = \frac{p\beta}{2}(r^{-2} + b^{-2}). \tag{4.21}$$

Figure 1 shows initial and steady state values of σ_θ.

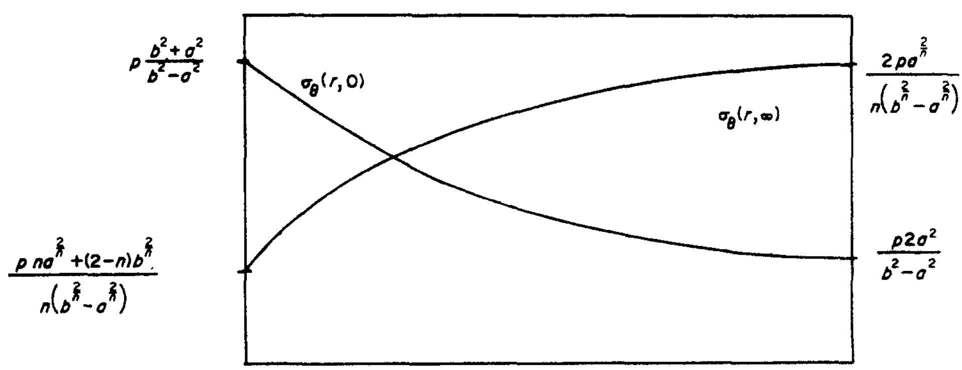

Fig. 1. Initial and steady state values of σ_θ.

In order to make some failure predictions, we shall use our bounds on effective stress to obtain bounds on the maximum principal stress. We will consider an incompressible cylindrical pressure vessel undergoing secondary creep subject to a constant internal pressure p.

It follows from (4.13) and (4.15) that the maximum principal stress at $r = a$ is

$$\sigma_{\max}(a, t) = \sigma_\theta(a, t) = \frac{2}{\sqrt{3}}(\sigma(a, t) - P). \tag{4.22}$$

The short time lower bound at $r = a$ is given by eqns (2.20), (2.21) as

$$\sigma(a, t) \geqslant \frac{\beta P}{a^2} - m(a)t, \tag{4.23}$$

where

$$m(a) = \frac{\mu\beta^n P^n}{a^2}(a^{2(1-n)} - b^{2(1-n)}). \tag{4.24}$$

Thus, by defining $\underline{\sigma}_{\max}(a, t)$ as

$$\underline{\sigma}_{\max}(a, t) = \frac{2}{\sqrt{3}}\left(\frac{\beta P}{a^2} - m(a)t - P\right), \tag{4.25}$$

we have

$$\sigma_{\max}((a, t) \geqslant \underline{\sigma}_{\max}(a, t), \tag{4.26}$$

provided failure has not occurred anywhere in the body prior to time t. This stipulation must be made because the field equations from which our bounds were derived do not hold in damaged subregions.

Let T_a be the solution of the equation

$$[\sigma_{cB}^{(5)}]^\nu 10^5 = \int_0^{T_a} \underline{\sigma}^\nu_{\max}(a, \tau)\,\mathrm{d}\tau, \tag{4.27}$$

if it exists. Then, by (4.4),

$$\int_0^{t_a} \sigma^\nu_{\max}(a, \tau)\,\mathrm{d}\tau = \int_0^{T_a} \underline{\sigma}^\nu_{\max}(a, \tau)\,\mathrm{d}\tau, \tag{4.28}$$

where t_a is the time at which failure occurs at the inner surface. From physical considerations $\sigma_{\max}$ is always positive. If we assume that we are only dealing with times at which $\underline{\sigma}_{\max}$ is positive, and that failure has not been initiated at a point other than the inner surface, it is apparent from eqns (4.26) and (4.28) that

$$t_a \leqslant T_a.$$

The number T_a can be interpreted in the following way: If the first point of failure in the entire body is at the inner surface, it will occur before time T_a. If failure is initiated elsewhere, T_a is nevertheless, an upper bound for the starting time of failure for the entire body.

To find T_a explicitly we integrate eqn (4.27) using (4.25) to get

$$T_a = \frac{1}{m(a)}\left\{\frac{\beta P}{a^2} - P - \left(\left(\frac{\beta P}{a^2} - P\right)^{\nu+1} - (\nu+1)m(a)10^5\left[\frac{\sqrt{3}}{2}\sigma_{cB}^{(5)}\right]^\nu\right)^{(1/\nu+1)}\right\}. \tag{4.29}$$

Using the definitions

$$R \equiv \left(\frac{a}{b}\right)^2, \tag{4.30}$$

$$\lambda \equiv \mu(\nu+1)10^5\left[\frac{\sqrt{3}}{2}\sigma_{cB}^{(5)}\right]^{\nu}, \tag{4.31}$$

$$k \equiv n-\nu-1, \tag{4.32}$$

and (4.24), we may rewrite (4.29) in the form

$$T_a = \frac{1}{2\mu(1-R^{n-1})}\left(\frac{1-R}{2P}\right)^{n-1}\left(1+R-\left((1+R)^{\nu+1}-\frac{2^n\lambda P^k(1-R^{n-1})}{(1-R)^k}\right)^{1/(\nu+1)}\right). \tag{4.33}$$

The condition that $\sigma_{max}(a,t)$ be positive can, from eqn (4.25), be stated as

$$\frac{\beta P}{a^2}-m(a)T_a-P\geq 0, \tag{4.34}$$

or

$$T_a \leq \frac{\frac{\beta P}{a^2}-P}{m(a)}. \tag{4.35}$$

Using (4.24) and (4.30) this condition becomes

$$T_a \leq \frac{(1+R)(1-R)^{n-1}}{\mu 2^n P^{n-1}(1-R^{n-1})}. \tag{4.36}$$

Substituting (4.33) into (4.36) and simplifying, we find that the integrand in (4.27) is positive provided

$$(1+R)^{n-k}(1-R)^k-\lambda 2^n P^k(1-R^{n-1})\geq 0. \tag{4.37}$$

Now let us consider the outer surface of the vessel. From eqn (4.15) with $r=b$ we have

$$\sigma_{max}(b,t)=\sigma_\theta(b,t)=\frac{2}{\sqrt{3}}\sigma(b,t). \tag{4.38}$$

The short time lower bound at $r=b$ is given by (2.22) as

$$\sigma(b,t)\leq\frac{\beta P}{b^2}+m(b)t. \tag{4.39}$$

Thus, if we let

$$\bar{\sigma}_{max}(b,t)=\frac{2}{\sqrt{3}}\left(\frac{\beta P}{b^2}+m(b)t\right), \tag{4.40}$$

we get

$$\sigma_{max}(b,t)\leq\bar{\sigma}_{max}(b,t). \tag{4.41}$$

Defining T_b as the solution of

$$\sigma_{cB}^{(5)\nu}10^5=\left(\frac{2}{\sqrt{3}}\right)^{\nu}\int_0^{T_b}\left(\frac{\beta P}{b^2}+m(b)\tau\right)^{\nu}\mathrm{d}\tau, \tag{4.42}$$

we find by comparison with eqn (4.4) that

$$\int_0^{t_b}\sigma_{max}^{\nu}(b,\tau)\,\mathrm{d}\tau=\int_0^{T_b}\bar{\sigma}_{max}^{\nu}(b,\tau)\,\mathrm{d}\tau, \tag{4.43}$$

where t_b is the time at which failure is initiated at b. By (4.41) and (4.43),

$$t_b \geq T_b. \tag{4.44}$$

Thus *if failure is initiated at the outer surface it will be at a time greater than* T_b. If failure is initiated elsewhere, no conclusions can be drawn.

T_b is computed from (4.42) to be

$$T_b = \frac{1}{m(b)}\left\{\left[\left(\frac{\sqrt{3}}{2}\sigma_{cB}^{(5)}\right)^{\nu} 10^5(\nu+1)m(b) + \left(\frac{\beta P}{b^2}\right)^{\nu+1}\right]^{1/(\nu+1)} - \frac{\beta P}{b^2}\right\}. \tag{4.45}$$

Using eqns (4.30)–(4.32) and (2.21) with $r = b$, we can rewrite this as

$$T_b = \frac{1}{\mu(1-R^{n-1})}\left(\frac{1-R}{2P}\right)^{n-1}\left[\left(1 + \frac{\lambda(2P)^k(1-R^{n-1})}{(1-R)^k R^{\nu}}\right)^{1/(\nu+1)} - 1\right]. \tag{4.46}$$

Next consider the condition

$$T_a < T_b. \tag{4.47}$$

If this inequality holds, the following interpretations can be given: (i) Failure will initiate in the body at some point other than the outer surface. (ii) Failure at the inner surface will precede failure at the outer surface provided it did not previously start at an interior point. Again the above qualifications are necessary since, once failure is initiated in a subregion, the field equations change.

Using (4.29) and (4.46), we can put condition (4.47) in the form

$$3 + R \leq \left[(1+R)^{\nu+1} - \frac{\lambda 2^n P^k (1-R^{n-1})}{(1-R)^k}\right]^{1/(\nu+1)} + 2\left[1 + \frac{\lambda}{R^{\nu}}\left(\frac{2P}{1-R}\right)^k (1-R^{n-1})\right]^{1/(\nu+1)}. \tag{4.48}$$

These results can be stated as follows:

Theorem 4.1. *Consider an incompressible cylindrical pressure vessel undergoing secondary creep and subject to a constant internal pressure* p. *Then, if failure is initiated at the outer surface, it will be at a time greater than* T_b *given by* (4.46). *If* (4.37) *is satisfied and damage is initiated at the inner surface, it will occur before a time* T_a *given by eqn* (4.33). *In any event, failure will begin somewhere in the body before time* T_a. *Also, inequality* (4.48) *being satisfied guarantees that failure will initiate somewhere other than the outer surface.*

As a special case of Theorem 4.1, consider $k = 0$. This implies that $\nu = n - 1$. This is not unreasonable since Odqvist in[8] has determined that

$$0.62n < \nu < n, \tag{4.49}$$

and, for most metals,

$$\nu \approx 0.7n. \tag{4.50}$$

Also in[4] examples of metals were given where indeed $\nu = n - 1$. With this assumption, condition (4.37) may be written as

$$(1+R)^n - \lambda 2^n (1-R^{n-1}) \geq 0, \tag{4.51}$$

and condition (4.48) becomes

$$[(1+R)^n - \lambda 2^n(1-R^{n-1})]^{(1/n)} + 2\left[1 - \lambda + \frac{\lambda}{R^{n-1}}\right]^{(1/n)} \geq 3 + R. \tag{4.52}$$

Note that both (4.51) and (4.52) are independent of pressure. That is, we can predict when failure will occur at a place other than the outer surface no matter what the internal pressure is. Also, since the function

$$f(R) = (1+R)^n - \lambda 2^n (1 - R^{n-1})$$

is an increasing function of R, for $R \geq 0$, we see that (4.51) is automatically satisfied if

$$\lambda \leq \frac{1}{2^n}. \tag{4.53}$$

Thus, failure will be initiated at the inner surface or an interior point for all R satisfying

$$0 < R < R_0, \tag{4.54}$$

where R_0 is the smallest positive root of

$$0 = [(1+R)^n - \lambda 2^n (1 - R^{n-1})]^{(1/n)} + 2\left[1 - \lambda + \frac{\lambda}{R^{n-1}}\right]^{(1/n)} - 3 - R. \tag{4.55}$$

For physical situations in which considerable stress redistribution has occurred prior to the onset of damage, it is natural to use the long-time bounds derived in Section 3 and to assume that damage is initiated at the outer surface. To this end, we recall the inequality (3.22) which has the form

$$r^j |\sigma^n(r,t) - \sigma^n(r,\infty)| \leq (A_1 + A_2 t)\, e^{-Ct}, \tag{4.56}$$

where

$$A_1 = |v(0) - lv(0)| + R^{-n+1} \| lv - v \|(0), \tag{4.57}$$

$$A_2 = CR^{-n+1} \| lv - v \|(0), \; v = r^j \sigma^n. \tag{4.58}$$

Applying (3.23) to the left side of (4.56), we get

$$r^j |\sigma^\nu(r,t) - \sigma^\nu(r,\infty)| \leq \frac{\nu}{n} \left(\frac{\beta P}{b^j}\right)^{\nu - n} (A_1 + A_2 t)\, e^{-Ct}. \tag{4.59}$$

Here we have also used the fact that

$$\sigma(r,t) \geq \frac{\beta P}{b^j}. \qquad (a \leq r \leq b, t > 0)$$

Thus, for $r = b$, (4.59) implies the inequalities

$$\sigma^\nu(b,\infty) - (L_1 + L_2 t)\, e^{-Ct} \leq \sigma^\nu(b,t) \leq \sigma^\nu(b,\infty) + (L_1 + L_2 t)\, e^{-Ct}, \tag{4.60}$$

where

$$L_i = \frac{\nu}{n} (\beta P)^{\nu - n} b^{j(n - \nu - 1)} A_i \qquad (i = 1, 2). \tag{4.61}$$

By (4.4) and the fact that

$$\sigma_{\max}(b,t) = \sigma_\theta(b,t) = \left(\frac{2}{\sqrt{3}}\right)^{3-j} \sigma(b,t) \tag{4.62}$$

(see 4.15, 4.17), it follows that the initial damage time, t_b, satisfies

$$\lambda = \int_0^{t_b} \sigma^\nu(b,t)\, dt, \tag{4.63}$$

where

$$\lambda = 10^5(\sigma_{cB}^{(5)})^\nu \left(\frac{\sqrt{3}}{2}\right)^{(3-j)\nu}. \tag{4.64}$$

Define T_1, T_2 as the solutions of the equations

$$\lambda = \int_0^{T_j} [\sigma^\nu(b,\infty) + (-1)^j(L_1 + L_2 t)\,e^{-Ct}]\,dt \qquad (j = 1, 2). \tag{4.65}$$

Then, by virtue of (4.60) and (4.63), we have

$$T_2 \leqslant t_b \leqslant T_1, \tag{4.66}$$

provided that damage is initiated in the body at $r = b$. After integration, (4.65) becomes

$$\lambda = \sigma^\nu(b,\infty)T_j + \frac{(-1)^{j+1}}{C}\left[\left(L_1 + \frac{L_2}{C}\right)(e^{-CT_j} - 1) + T_j L_2\, e^{-CT_j}\right]. \tag{4.67}$$

Since all of the constants in (4.67) are known *a priori*, T_j may be computed from it by a simple iteration scheme. This equation also yields bounds on T_j. In fact, since the L_i are positive, (4.67) implies

$$T_1 \leqslant \sigma^{-\nu}(b,\infty)\left[\lambda + \frac{(L_1 C + L_2)}{C^2}\right] \equiv T_U, \tag{4.68}$$

$$T_2 \geqslant \sigma^{-\nu}(b,\infty)\left[\lambda - \frac{(L_1 C + L_2)}{C^2}\right] \equiv T_L. \tag{4.69}$$

A measure of the relative error involved in the use of T_U and T_L is given by

$$\frac{T_U - T_L}{T_L} = \frac{2(L_1 C + L_2)}{\lambda C^2 - L_1 C - L_2}. \tag{4.70}$$

These results can be summarized by

Theorem 4.2. *Consider a symmetric pressure vessel with constant internal pressure* p. *Then, assuming failure is initiated at the outer surface, an upper bound on the time until brittle creep rupture is given by* T_U (4.68), *a lower bound is given by* T_L (4.69), *and the relative error between them is given by* (4.70).

Since these estimates were made using the long time estimates on effective stress, their accuracy is improved if conditions favor a quick redistribution of stress, i.e. high pressure, high temperature, a high creep constant, and a relatively thin vessel. Tables 4–6 give values of T_L and T_U from eqns (4.68) and (4.69) for various values of pressure and thickness. For $(b/a) = 1.1$ accuracy of the estimates is excellent, while for $(b/a) = 2.0$ accuracy was best for the low alloy steel and poor for stainless steel. Also for the stainless steel, accuracy was unexpectedly reduced as temperature increased. This is due to the fact that the creep rupture constant, $\sigma_{cB}^{(5)}$, decreases rapidly with increasing temperature.

In [3], Kachanov defines T_l, the time of latent failure, as the time of initiation of damage for the whole body which he computes using the steady state solution at $r = b$. Thus, in the notation of (4.63),

$$\lambda = \int_0^{T_l} \sigma^\nu(b,\infty)\,dt = T_l \sigma^\nu(b,\infty),$$

so that by (4.68) and (4.69),

$$T_l = \lambda\sigma^{-\nu}(b,\infty) = \frac{1}{2}(T_U + T_L).$$

Table 4. Values of T_L and T_U for an incompressible cylinder pressure = 5 kg/mm²

				$\frac{b}{a}$ = 1.1			$\frac{b}{a}$ = 1.5			$\frac{b}{a}$ = 2.0		
Material	Temp °C	n	ν	T_L	T_U	$\frac{T_U-T_L}{T_L}$	T_L	T_U	$\frac{T_U-T_L}{T_L}$	T_L	T_U	$\frac{T_U-T_L}{T_L}$
Low alloy steel (rolled)	550	4.15	3	1634	1634.6	.0003	155617	160705	.033	3.1×10^5	1.65×10^6	4.36
	600	2.74	2	399	399.4	.0016	9040	9200	.018	31494	35193	.117
	650	2.1	1.5	370.5	370.9	.0010	4085	4106	.005	11362	11528	.015
Carbon steel (rolled)	450	5	3.5	148	148.5	.006	0	6951	-	0	4.9×10^7	-
	500	3.3	2.3	286	288	.007	8950	11058	.24	0	1.08×10^5	-
	550	2.5	1	5352	5353	.0002	25803	25964	.0062	48574	51552	.061
Stainless steel (rolled)	500	5.6	3.9	3008	3012	.0014	1.6×10^5	1.97×10^6	11.4	0	3.5×10^9	-
	600	4.5	3.1	626	628	.002	52223	86465	.66	0	9.2×10^6	-
	650	4.0	2.8	245	249	.015	0	46633	-	0	1.24×10^7	-

Values of n and ν from Odqvist[4].

Table 5. Values of T_L and T_U for an incompressible cylinder pressure = 10 kg/mm²

				$\frac{b}{a}$ = 1.1			$\frac{b}{a}$ = 1.5			$\frac{b}{a}$ = 2.0		
Material	Temp °C	n	ν	T_L	T_U	$\frac{T_U-T_L}{T_L}$	T_L	T_U	$\frac{T_U-T_L}{T_L}$	T_L	T_U	$\frac{T_U-T_L}{T_L}$
Low alloy steel (rolled)	550	4.15	3	204.25	204.32	.0003	19484	20056	.029	37417	198587	3.24
	600	2.74	2	99.6	99.9	.0019	2256	2304	.021	7782	8889	.142
	650	2.1	1.5	131	131.2	.0013	1443	1453	.0066	4008	4085	.019
Carbon steel (rolled)	450	5	3.5	13.05	13.11	.0043	89	5093	56	0	3.07×10^6	-
	500	3.3	2.3	58	58.4	.007	1817	2245	.24	0	2.18×10^4	-
	550	2.5	1	2676	2676	.0001	12913	12970	.0044	24505	25558	.043
Stainless steel (rolled)	500	5.6	3.9	202	202	.0008	33875	108400	2.2	0	1.3×10^8	-
	600	4.5	3.1	73	73.2	.0017	6572	9603	.46	0	8.26×10^5	-
	650	4.0	2.8	35.3	35.7	.011	0	5685	-	0	1.41×10^6	-

Values of n and ν from Odqvist[4].

Table 6. Values of T_L and T_U for an incompressible cylinder pressure = 20 kg/mm²

				$\frac{b}{a}$ = 1.1			$\frac{b}{a}$ = 1.5			$\frac{b}{a}$ = 2.0		
Material	Temp °C	n	ν	T_L	T_U	$\frac{T_U-T_L}{T_L}$	T_L	T_U	$\frac{T_U-T_L}{T_L}$	T_L	T_U	$\frac{T_U-T_L}{T_L}$
Low alloy steel (rolled)	550	4.15	3	25.53	25.54	.0003	2439	2504	.026	6790	23890	2.52
	600	2.74	2	24.9	24.97	.002	563	577	.025	1918	2250	.173
	650	2.1	1.5	46.3	46.4	.002	510	514	.0087	1413	1449	.026
Carbon steel (rolled)	450	5	3.5	1.15	1.158	.003	72.5	386	4.3	0	1.92×10^5	-
	500	3.3	2.3	11.8	11.9	.007	369	456	.24	0	4434	-
	550	2.5	1	1338	1338	$9.1\ 10^{-5}$	6461	6481	.003	12.30	12330	.03
Stainless steel (rolled)	500	5.6	3.9	13.5	13.5	.0005	3229	6301	.95	0	5.5×10^6	-
	600	4.5	3.1	8.52	8.53	.0013	810	1077	.33	0	7.43×10^4	-
	650	4.0	2.8	5.07	5.12	.009	25	707	27	0	1.5×10^5	-

Values of n and ν from Odqvist[4].

It is also clear that for $(L_1C + L_2)C^{-2}$ small, use of the steady state solution $\sigma(r, \infty)$ gives a good estimate for the time of latent failure. This turns out to be the case when $b/a = 1.1$.

5. CONCLUSIONS

The investigations presented above were undertaken in order to develop the idea, suggested in[1], that the bounds (1.4) established in that paper might be employed in the derivation of more refined bounds. In Section 2 of the present paper, this was achieved by using the bounds of[1] to obtain new bounds which reduce to the exact solution as $t \to 0$. Thus, accuracy for small times is greatly enhanced. The price paid for this is that, after a critical time t^*, the new bounds become less accurate than (1.4). However, Tables 1 and 2 reveal that, in various situations, t^* can be quite large.

In[2], an argument was presented for the uniform convergence as $t \to \infty$ of the transient secondary creep solution to the formally derived steady-state solution (3.1). Section 3 of the present paper was devoted to a reworking of the analysis of[2] in order to obtain, in addition to this convergence result, explicit bounds for the difference between $\sigma(r, t)$ and $\sigma(r, \infty)$. The modified analysis also eliminated the need for a Sobolov-type inequality in the derivation of the pointwise bound. In a situation analogous to that of Section 2, the present bounds become inferior to (1.4) *prior* to some other critical time which has also been denoted t^*. Table 3 furnishes examples in which t^* is quite small.

Thus for secondary creep, three distinct types of bounds are available. The short term bounds of Section 2, the intermediate bounds derived in[2], and the long-term bounds of Section 3. As is suggested by the accompanying tables, for certain combinations of material, temperature and thickness, fewer bounds may be required. In the case of primary creep, the situation is less satisfactory, in that we are presently unable to establish longtime bounds.

Section 4 furnishes one possible application of the stress bounds, namely, to the estimation of the time and locus of initial damage, according to the damage theory of Kachanov[3]. The latter has been used, mainly because our stress bounds can be combined with it very readily to produce damage estimates. Other more complicated damage theories exist, such as that of Rabotnov[9]† which include the "coupling" effect of damage on the stress distribution. For that range of circumstances in which this effect is significant our results would not apply. A theoretical investigation of those circumstances under which Kachanov's predictions furnish a good approximation of Rabotnov's is beyond the scope of this paper. About the only practical justification we can give for use of an "uncoupled" theory is that it does, to some extent, agree with long-standing engineering practice, as is asserted by Rabotnov himself ([9], p. 344).

Note added in proof. Since this paper was written, the authors have found additional intermediate bounds for secondary creep (see, e.g. [10]).

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